Lecture Notes in Computer Science

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Group Theoretical Methods in Image Processing

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Foreword

These Lecture Notes are based on a series of lectures I gave at the Linköping University Department of Electrical Engineering in 1988. In these lectures I tried to give an overview of the theory of representation of compact groups and some applications in the fields of image science and pattern recognition.

The participants in the course had a masters degree in electrical engineering and no deeper knowledge in group theory or functional analysis. The first chapter is thus used to introduce some basic concepts from (algebraic) group theory, topology, functional analysis and topological groups. This chapter contains also some of the main results from these fields that are used in the following chapters. This chapter cannot replace full courses in algebra, topology or functional analysis but it should give the reader an impression of the main concepts, tools and results of these branches of mathematics. Some of the groups that will be studied more extensively in the following chapters are also introduced in this chapter.

The central idea of group representations is then introduced in Chapter 3. The basic theme in this chapter is the study of function spaces that are invariant under a group of transformations such as the group of rotations. It will be demonstrated how the algebraic and topological properties of the group determine the structure of these invariant spaces: if the transformation group is commutative, it will be shown that the minimal, invariant spaces are all one-dimensional and if the group is compact, then these spaces are all finite-dimensional. This result explains, for example, the unique role of the complex exponential function. In this chapter we concentrate on the derivation of qualitative information about the invariant function spaces but (with the exception of the complex exponential function that is connected to the group of real numbers $\mathbf{R}$ and the group of 2-D rotations) no such function spaces are actually constructed. The construction of these invariant function spaces for some important groups is demonstrated in Chapter 4. There we use methods from the theory of Lie groups to find the functions connected to certain important transformation groups. These groups include the scaling group, the 2-D and 3-D rotation groups, the group of rigid motions and the special linear group of the 2×2 matrices with complex entries and determinant equal to one.

In Chapter 5 we conclude the mathematical part of these lectures by demonstrating how the concept of a Fourier series can be generalized. The basic idea behind this generalization is the following: Consider a periodic function $f(\phi)$ with period 2π. Usually we think of such a function as a (complex-valued) function on an interval or a circle. In the new interpretation we view ϕ as the angle of a rotation and f becomes thus a function defined on the group of 2-D rotations. The decomposition of f into a Fourier series means thus that a function on the group of 2-D rotations can be decomposed into a series of complex exponentials. In Chapter 5 we demonstrate that a similar decomposition is possible for functions defined on compact groups.

In Chapter 6 it is then demonstrated how the results from the previous chapters can be used to investigate and solve problems in image science. Our approach to solve these problems is based on the observation that many problems in image science are highly regular or symmetric. We investigate how these symmetries can be used to design problem solving strategies that can make use of this information.

First it is demonstrated how one can find eigenvectors of operators that commute with the group transformation. As important examples it is shown that eigenvalue problems connected to the group of 2-D rotations lead to the complex exponentials as solutions, whereas operators that commute with 3-D rotations have eigenvectors connected to the spherical harmonics. Examples of such operators are the Laplacian and the Fourier Transform.

As an example we then give a short overview of series expansions methods in computer aided tomography. Here we use the fact that the scanning process is rotationally symmetric. The functions used in the reconstruction process are thus tightly coupled to the rotation groups.

Another application deals with the generalization of the well-known edge-detection problem. We generalize it to the case where the interesting pattern is an arbitrary function and where the images are of dimension two or higher. We only assume that the transformation rule can be defined in a group theoretical sense. Under these conditions we describe a method that allows us to solve the pattern detection problem. Moreover we will show that the method derived is optimal in a certain sense. It will also be demonstrated why the derived filter functions are often found as solutions to optimality problems and why they have a number of other nice properties. In the rotational case for example it will be shown that the filter functions are eigenfunctions of the Laplacian and the Fourier Transform. Another result shows that group representations naturally turn up in the Karhunen Loeve expansion which makes these results interesting for applications in image coding.

We will see that most of the problems mentioned can be formulated in terms of so called compact groups. These are groups that do not have too many elements. We will also, briefly, study the easiest example of a non-compact, non-commutative group. This is the group of all complex 2×2 matrices with determinant one. It will be demonstrated how this group can be used in the analysis of camera motion.

In the last example we study some problems from the theory of neural networks. Here one investigates so-called basic units. These basic units are a model of one or a small number of nerve-cells. One approach to investigate and design such basic units is to view such a unit as a device that correlates the incoming signal with its internal state. The correlation result is then send to other units in the network and the internal state is updated according to some rule. In the last section we assume that the unit updates its internal state in such a way that the computed result is maximal in a mean-squared sense. We then investigate what happens when the basic unit is trained with a set of input functions that are transformed versions of a fixed pattern. As example we consider the cases were the input signals are rotated edge or line patterns. We show that in these cases the stable states of the unit are connected to the representations of the group of transformations that generated the input patterns from the fixed pattern.

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